Chapter 2

Relativistic Kinematics

In this chapter, the framework of Einstein's special theory of relativity is presented. Transformations that render the spacetime interval $-c^2t^2 + x^2 + y^2 + z^2$ unchanged define the category of physical theories that are Lorentz invariant. Further relativistic invariants, used to transform particle and photon distributions, are derived. The kinetic theory of reaction rates and secondary spectra occupies the second half of this chapter.

2.1 Lorentz Transformation Equations

Consider two coordinate systems K and K' in uniform relative motion, defining an inertial reference system. The reference frames are aligned along the $\hat{x}$ and $\hat{x}'$ axes, with frame K' moving at speed $v = \beta c$ in the positive $\hat{x}$ direction with respect to frame K (Fig. 2.1). Assume that a ruler and a clock are used to measure location and time in each frame. According to the postulates of special relativity, the laws of physics are the same in inertial reference systems, and the speed of light c is the same in both frames. To satisfy these conditions requires that the interval

$$-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2 = 0 \ . \tag{2.1}$$

Assuming homogeneity and isotropy of space, one can easily show that the Lorentz transformation equations are the simplest linear equations satisfying eq. (2.1) that connect location $\vec{x}$ at time t measured in K with location $\vec{x}'$ at time t' measured in K'. They are

$$
\begin{aligned}
t' &= \Gamma(t - \beta x/c) \\
y' &= y \\
z' &= z \\
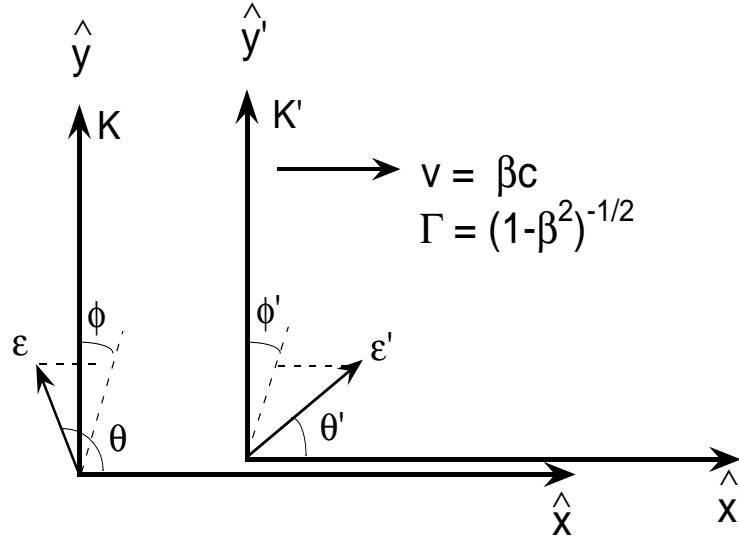x' &= \Gamma(x - \beta ct) \ ,
\end{aligned}
\tag{2.2}
$$

Figure 2.1: Bulk frame K' moving with speed $v = \beta c$ in the $\hat{x}$ direction with respect to frame K. Direction of a photon or relativistic particle with dimensionless energy ϵ makes an angle θ with respect to the $\hat{x}$ axis. The azimuth ϕ is the angle between the projection of this vector on the $\hat{x}$-$\hat{y}$ plane and the $\hat{y}$ axis, and similarly for primed quantities in the K' frame.

where $\Gamma = 1/\sqrt{1 - \beta^2}$. The reverse transformations are

$$
\begin{aligned}
t &= \Gamma(t' + \beta x'/c) \\
y &= y' \\
z &= z' \\
x &= \Gamma(x' + \beta ct') \ .
\end{aligned}
\tag{2.3}
$$

The set of four numbers $x^\mu = (ct, \vec{x}) = (ct, x, y, z)$ defines a spacetime event. Events in two inertial reference frames are related by eqs. (2.2) and (2.3).

Suppose the length of an object is measured in frame K. At the same time t, one measures length $\Delta x = x_2 - x_1$ of an object in motion. In the K' frame, $x_2' = \Gamma(x_2 - \beta ct)$ and $x_1' = \Gamma(x_1 - \beta ct)$, from eq. (2.2), so that $\Delta x' = x_2' - x_1' = \Gamma \Delta x$. Hence $\Delta x = \Delta x'/\Gamma$, so that

$$
dx = \frac{dx'}{\Gamma} \ .
\tag{2.4}
$$

The length of a moving object measured along its direction of motion is shorter than its length as measured in the proper frame of the object. This is the phenomenon of length contraction.

Now consider a clock at rest in the K' coordinate system, so x' remains constant in K'. The relationship between the time measured in the stationary frame K to that in the comoving frame K' is, from eq. (2.3), simply $\Delta t = t_2 - t_1 = \Gamma(t_2' + \beta x'/c) - \Gamma(t_1' + \beta x'/c) = \Gamma(t_2' - t_1') = \Delta t'$, so that

$$dt = \Gamma dt' \ . \tag{2.5}$$

This is the phenomenon of time dilation.

In the general case, the K' frame does not travel along the $\hat{x}$ axis. The Lorentz transformation equations for the coordinate $r_\parallel$ along the direction of motion and the coordinate $r_\perp$ transverse to the direction of motion can then be written as

$$t' = \Gamma(t - vr_\parallel/c) \ , \ r_\parallel' = \Gamma(r_\parallel - vt) \ , \ \text{ and } \ r_\perp' = r_\perp \ . \tag{2.6}$$

Here $\theta = \arccos\mu$ is the angle between the direction of motion of the K' system and the $\hat{x}$ axis of frame K, so that $r_\parallel = (\vec{v} \cdot \vec{r})/v$, $r_\perp = \sqrt{1 - \mu^2}|\vec{r}|$, and $vr_\parallel = \vec{v} \cdot \vec{r} = vr\mu$. The reverse transformation equations are

$$t = \Gamma(t' + vr_\parallel'/c) \ , \ r_\parallel = \Gamma(r_\parallel' + vt') \ , \ \text{ and } \ r_\perp = r_\perp' \ . \tag{2.7}$$

The invariance of the interval, eq. (2.1), also implies that the spacetime interval

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$$

is invariant. Consider a particle at rest in the origin of K'. Therefore

$$dt'^2 = dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}\right) = dt^2(1 - v^2/c^2) \ .$$

Thus the proper time

$$dt' == \frac{dt}{\Gamma} = \frac{dt}{\gamma} \tag{2.8}$$

as measured in the rest frame of a particle is invariant because, in this case, the particle Lorentz factor γ equals the bulk Lorentz factor Γ. Hence $dt = \gamma dt'$, in agreement with the time dilation formula, eq. (2.5).

2.2 Four Vectors and Momentum

The four-vector spacetime coordinate $x^\mu = (ct, \vec{x}) = (x^0, x^1, x^2, x^3)$ transforms according to the Lorentz transformations, eqs. (2.2) and (2.3). A four vector is defined as a set of four quantities that transform according to eq. (2.9). Thus

$$\begin{aligned} x'^0 &= \Gamma(x^0 - \beta x^1) \\ x'^1 &= \Gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \ . \end{aligned} \tag{2.9}$$

Four vectors can be constructed from the spacetime four vector and invariants that are unchanged by Lorentz transformations. The four-vector momentum

$$p^\mu = -mc\frac{dx^\mu}{ds} = mc\gamma(1, \vec{\beta}_{par}) = mc(\gamma, \vec{p}_{par}),\qquad(2.10)$$

where $\vec{\beta}_{par} = d\vec{x}/dt$ and $\vec{p}_{par} = \vec{\beta}_{par}\gamma$, and we use the invariant $ds = -cdt' = -cdt/\gamma$, eq. (2.8) associated with the proper time of a particle moving with Lorentz factor $\gamma = 1/\sqrt{1 - \beta_{par}^2}$ in the K frame. The time component of eq. (2.10) is equal to E/c, where the total particle energy

$$E = \gamma mc^2 \ .$$

The quantity m is the invariant particle rest mass.

Because eq. (2.10) is a four vector, it transforms according to eq. (2.9). Thus one obtains the Lorentz transformation equations

$$
\begin{aligned}
\gamma' &= \Gamma(\gamma - \beta p_x) = \Gamma\gamma(1 - \beta\beta_{par,x}) \\
p'_x &= \Gamma(p_x - \beta\gamma)\ ,\ \text{or}\ \gamma'\beta'_{par,x} = \gamma\Gamma(\beta_{par,x} - \beta) \\
p'_x &= p_y \\
p'_z &= p_z\ ,
\end{aligned}
\qquad(2.11)
$$

for particle Lorentz factor and dimensionless momentum, with the reverse transformation obtained by letting $\beta \to -\beta$ and switching primed and unprimed quantities. These equations can be derived, in analogy with the Lorentz transformation equations for the spacetime event, from the invariance of $-(mc)^2 = -(E/c)^2 + (mcp)^2 = -(mc)^2(\gamma^2 - \beta_{par}^2\gamma^2)$.

Because the x-component of dimensionless momentum can be written as $p_x = \gamma\beta_{par,x} = \gamma\beta_{par}\mu$, where $\theta = \arccos\mu$ is the angle between the direction between the particle momentum and the $\hat{x}$ axis,

$$\gamma' = \Gamma\gamma(1 - \beta\beta_{par}\mu)\ ,\qquad(2.12)$$

and

$$\beta'_{par}\gamma'\mu' = \Gamma\gamma(\beta_{par}\mu - \beta)\ .\qquad(2.13)$$

The ratio of eqs. (2.13) and (2.12) is

$$\beta'_{par}\mu' = \frac{\beta_{par}\mu - \beta}{1 - \beta_{par}\beta\mu}\ .\qquad(2.14)$$

For massless photons or highly relativistic particles with $\beta_{par} \to 1$ and $\gamma \gg 1$, we let $\gamma \to \epsilon$. Thus

$$
\begin{aligned}
\epsilon' &= \Gamma\epsilon(1 - \beta\mu)\ , & (2.15) \\
\mu' &= \frac{\mu - \beta}{1 - \beta\mu}\ ,\ \text{and} & (2.16) \\
\phi' &= \phi\ , & (2.17)
\end{aligned}
$$

now writing the energy in terms of cosine angle μ and azimuth angle ϕ (Fig 2.1). The reverse transformation equations for photons and relativistic particles are

$$\epsilon = \Gamma\epsilon'(1 + \beta\mu') \, , \tag{2.18}$$

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \, , \text{ and} \tag{2.19}$$

$$\phi = \phi' \, . \tag{2.20}$$

Eqs. (2.15) − (2.20) can be derived for photons by considering the photon four-vector momentum $k^\mu = (\hbar/c)(\omega, \vec{k})$, with $\epsilon = h\nu/m_e c^2 = \hbar\omega/m_e c^2$.

If a photon in the bulk comoving frame is emitted at right angles to the direction of motion, then $\theta' = \pi/2$ and $\mu' = 0$. The cosine angle of the photon in frame K is $\mu = \beta$, from eq. (2.19). For highly relativistic bulk speeds, $\Gamma \gg 1$ and $\beta \approx 1 - (1/2\Gamma^2) \approx 1 - (\theta^2/2)$. All photons emitted in the forward direction in K' are therefore beamed into a narrow range of angles $\theta \gtrsim 1/\Gamma$ in K. This illustrates the phenomenon of relativistic beaming.

2.3 Relativistic Doppler Factor

Eq. (2.15) shows that the photon energy in frame K is related to the photon energy in frame K' according to the relation

$$\frac{\epsilon}{\epsilon'} = \delta_{\mathrm{D}} \equiv [\Gamma(1 - \beta\mu)]^{-1} \, , \tag{2.21}$$

where δ_{D} is the Doppler factor. In the limit that of large bulk Lorentz factors and small observing angles along the line ofsight,

$$\delta_{\mathrm{D}} \overset{\Gamma \gg 1, \, \theta \ll 1}{\longrightarrow} \frac{2\Gamma}{1 + \Gamma^2\theta^2} \, , \tag{2.22}$$

It is useful to derive this factor by considering an observer receiving photons emitted at an angle θ with respect to the direction of motion of frame K' in the stationary frame K (Fig. 2.2). During time Δt, as measured in stationary frame K, the bulk system moves a distance

$$\Delta x = \beta c \Delta t = \beta\Gamma c \Delta t' \, ,$$

where the last expression relates the change in distance to the comoving time element using the time dilation formula, eq. (2.5).

A light pulse emitted at stationary frame time t and location x is received at observer time

$$t^{ob} = t + \frac{d}{c} - \frac{x\cos\theta}{c} \, , \tag{2.23}$$

where d is the distance of the observer from the origin of stationary frame K. At a later time $t + \Delta t$, a second pulse of light is emitted which is received by the observer at time

$$t^{ob} + \Delta t^{ob} = t + \Delta t + \frac{d}{c} - \frac{(x + \Delta x)\cos\theta}{c} \, . \tag{2.24}$$
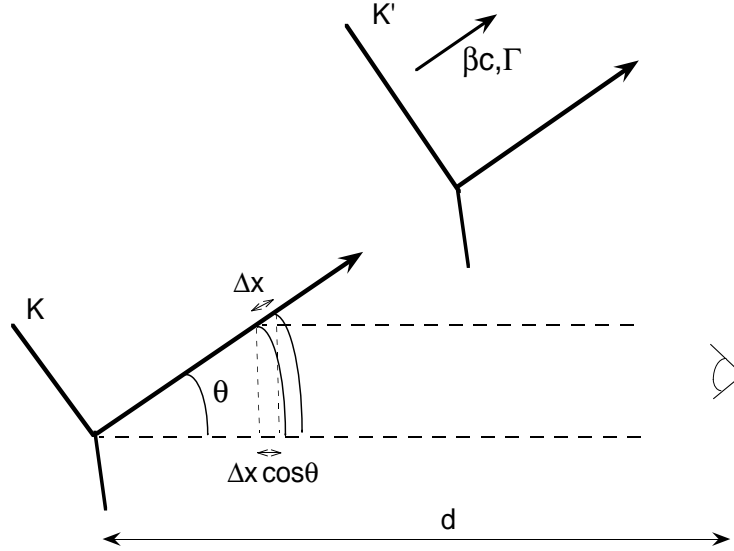
Figure 2.2: Geometry of the Doppler Effect.

Subtracting eq. (2.23) from eq. (2.24), and taking the limit of differential quantities gives

$$dt^{ob} = \frac{dx}{\beta c}\left(1 - \beta\cos\theta\right) = \Gamma dt'(1 - \beta\mu) = \frac{dt'}{\delta_{\mathrm{D}}} \ . \tag{2.25}$$

Because $\epsilon = h\nu/m_e c^2$ and $\nu \propto 1/\Delta t$,

$$\frac{dt'}{dt^{ob}} = \frac{\epsilon}{\epsilon'} \ , \tag{2.26}$$

and $\epsilon' = \epsilon/\delta_{\mathrm{D}}$, eq. (2.15.

2.4   Three Useful Invariants

The invariance of the four-volume $dtdV = dtd^3\vec{x}$ is demonstrated. Without loss of generality, align the coordinate axes along the direction of relative motion, as in Fig. 2.1. The quantity

$$dtdV = dtdxdydz \ = \ J\begin{pmatrix} t & x & y & z \\ t' & x' & y' & z' \end{pmatrix} dt'dx'dy'dz' \ , \tag{2.27}$$

where the Jacobian of the transformation, from eq. (2.3), is

$$J\begin{pmatrix} t & x & y & z \\ t' & x' & y' & z' \end{pmatrix} = \begin{vmatrix} \frac{\partial t}{\partial t'} & \frac{\partial t}{\partial x'} & \frac{\partial t}{\partial y'} & \frac{\partial t}{\partial z'} \\ \frac{\partial x}{\partial t'} & \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial t'} & \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial t'} & \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{vmatrix} =$$

$$\begin{vmatrix} \Gamma & \beta/c & 0 \\ \Gamma\beta c & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \Gamma^2(1 - \beta^2) = 1 . \tag{2.28}$$

Thus

$$dV\,dt = dV'\,dt' = inv . \tag{2.29}$$

We now examine the transformation quantities of the momentum volume element

$$d^3\vec{p'} = dp'_x dp'_y dp'_z = |\frac{\partial p'_x}{\partial p_x}|dp_x dp_y dp_z , \tag{2.30}$$

and show that the phase-space element $d^3\vec{p}/E$ is an invariant, noting that the perpendicular momentum components dp_y and dp_z are unchanged by a boost along the $\hat{x}$ axis. From eqs. (2.11) and (2.12),

$$|\frac{\partial p'_x}{\partial p_x}| = \Gamma - \beta\Gamma|\frac{\partial\gamma}{\partial p_x}| = \frac{\Gamma(\gamma - \beta p_x)}{\gamma} = \frac{\gamma'}{\gamma} = \frac{E'}{E} . \tag{2.31}$$

Note that $\gamma = \sqrt{1 + p^2} = \sqrt{1 + p_x^2 + p_y^2 + p_z^2}$, so that $\partial p_i/\partial\gamma = \gamma/p_i$ and $\partial p/\partial\gamma = \gamma/p$. Thus

$$\frac{d^3\vec{p}}{E} = \frac{d^3\vec{p'}}{E'} = inv , \tag{2.32}$$

and

$$\frac{d^3\vec{p}}{E} = \frac{p^2 dp d\Omega}{E} \rightarrow \epsilon d\epsilon d\Omega , \tag{2.33}$$

is invariant, with the final expression applying to photons and relativistic particles.

Finally, we establish the invariance of the phase-space volume $d\mathcal{V} = d^3\vec{x}d^3\vec{p}$. Consider particles or photons distributed in a small physical volume element $\Delta^3 x'$ and momentum volume element $\Delta^3 p'$. From eq. (2.29), $d^3\vec{x} = |dt'/dt|d^3\vec{x'} = d^3\vec{x'}/\Gamma$, which is just an expression of the length contraction formula, eq. (2.4). For a distribution of particles with a small spread of momenta in the K' frame, $|\vec{p'}| = \beta'\gamma' \ll 1$. Using the inverse transformation $p_x = \Gamma(p'_x + \beta\gamma')$ from eq. (2.11),

$$\frac{\partial p_x}{\partial p'_x} = \frac{\partial[\Gamma(p'_x + \beta\gamma')]}{\partial p'_x} = \Gamma + \beta\Gamma(\frac{\partial\gamma'}{\partial p'_x}) \rightarrow \Gamma ,$$

because $\partial\gamma'/\partial p'_x = p'_x/\Gamma \rightarrow 0$ under the stated conditions. Thus the phase space volume is an invariant. The invariance of $d\mathcal{V}$ also follows by noting that dt/E is the ratio of parallel 4-vectors [1].

2.5 Invariance of $u(\epsilon, \Omega)/\epsilon^3$, I_ϵ/ϵ^3, and $j(\epsilon, \Omega)/\epsilon^2$

The elementary invariants are the invariant 4-volume $d^3\vec{x}dt = dV\,dt$, the invariant phase-space element $d^3\vec{p}/E$, and the invariant phase volume $d\mathcal{V} = d^3\vec{x}d^3\vec{p}$.

Because the number N of particles or photons is invariant,

$$\frac{dN}{d\mathcal{V}} = \frac{1}{p^2}\,\frac{dN}{dV\,dp\,d\Omega} \rightarrow \frac{1}{\epsilon^2}\,\frac{dN}{dV\,d\epsilon\,d\Omega} = \frac{1}{m_e c^2 \epsilon^3}\,\frac{d\mathcal{E}}{dV\,d\epsilon\,d\Omega} \equiv \frac{1}{\epsilon^3}\,\frac{u(\epsilon,\Omega)}{m_e c^2}\;,\quad (2.34)$$

where the latter three expressions apply to photons and relativistic particles. Here we also introduce the quantity $\mathcal{E}$ to represent the total energy contained in a distribution of particles or photons, as distinct from the E or ϵ, which represents energy and dimenensionless energy, respectively, in a single particle or photon. The specific spectral energy density $u(\epsilon,\Omega)$ is defined so that $u(\epsilon,\Omega)dV\,d\epsilon d\Omega$ is the differential energy $d\mathcal{E}$ in particles or photons in differential volume dV with energy between ϵ and $\epsilon + d\epsilon$ that are directed into the solid angle element $d\Omega$ in the direction of Ω. From eq. (2.34), we see that

$$\frac{u(\epsilon,\Omega)}{\epsilon^3} = \frac{u'(\epsilon',\Omega')}{\epsilon'^3} = inv\;. \qquad (2.35)$$

A bundle of photons or relativistic particles directed into a differential solid angle interval $d\Omega$ sweep out a volume element $dV = cdtdA$ as they pass through a differential area element dA oriented along the direction Ω. Thus

$$u(\epsilon,\Omega) = \frac{d\mathcal{E}}{dV\,d\epsilon\,d\Omega} = \frac{1}{c}\,\frac{d\mathcal{E}}{dA\,dt\,d\epsilon\,d\Omega} = \frac{I_\epsilon(\Omega)}{c}\;.$$

Hence, $u(\epsilon,\Omega)/\epsilon^3$ and $I_\epsilon(\Omega)/\epsilon^3$ are invariants. The function $I_\epsilon(\Omega)$ is the intensity, and is considered in more detail in a later chapter.

The function

$$E\,\frac{dN}{d^3\vec{x}\,dt\,d^3\vec{p}} = \frac{1}{\epsilon^2}\,\frac{d\mathcal{E}}{dV\,dt\,d\epsilon\,d\Omega} = \frac{1}{\epsilon^2}\,j(\epsilon,\Omega)\;. \qquad (2.36)$$

is invariant, noting that the last two expressions apply to photons and relativistic particles. The function $j(\epsilon,\Omega)$ is the emissivity. A formalism to calculate emissivities is presented later in this chapter.

2.6 Relations between Transformed Quantities

Let $N(\epsilon,\Omega)d\epsilon d\Omega$ represent the differential number of photons or relativistic particles with energy between ϵ and $\epsilon + d\epsilon$ that are directed into differential solid angle interval $d\Omega$ in the direction Ω. Because the total number of photons or particles is invariant, so also is the quantity

$$\frac{dN}{\epsilon\,d\epsilon\,d\Omega} = \frac{N(\epsilon,\Omega)}{\epsilon}\;, \qquad (2.37)$$

using eq. (2.33). Thus $N(\epsilon,\Omega) = \delta_{\mathrm{D}}N'(\epsilon',\Omega')$, noting the definition, eq. (2.21), of the Doppler factor.
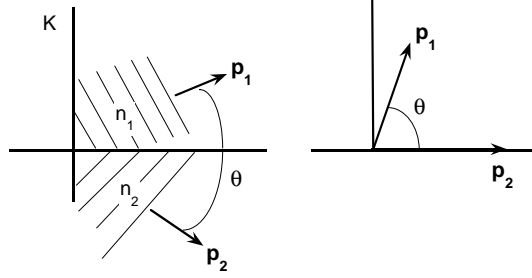
Figure 2.3: Sketch used to derive reaction rate between interacting particles or photons with stationary frame densities n_1 and n_2.

Consider the beaming pattern of radiation for an observer in frame K from a source that radiates isotropically in K'. Eq. (2.37) becomes

$$N(\epsilon, \Omega) = \delta_{\mathrm{D}} \frac{N'(\epsilon/\delta_{\mathrm{D}})}{4\pi} = \frac{1}{\Gamma(1 - \beta\mu)} \frac{N'[\Gamma\epsilon(1 - \beta\mu)]}{4\pi} \ . \qquad (2.38)$$

Furthermore, suppose that the spectrum of photons or particles is monochromatic in the comoving frame. In this case, $N'(\epsilon', \Omega') = N_0 \delta_{\mathrm{D}}(\epsilon' - \epsilon'_0)/4\pi$, and the total photon or particle energy in the comoving frame, in units of the electron rest mass, is $\mathcal{E}'_0 = N_0 \epsilon'_0$. The differential photon spectrum in the stationary frame is therefore $N(\epsilon, \Omega) = \delta_{\mathrm{D}} N_0 \delta(\epsilon/\delta_{\mathrm{D}} - \epsilon'_0)/4\pi = \delta^2_{\mathrm{D}} N_0 \delta(\epsilon - \delta_{\mathrm{D}} \epsilon'_0)/4\pi$, so that the total energy in frame K is

$$\mathcal{E} = \oint d\Omega \int_0^\infty d\epsilon \ \epsilon \ N(\epsilon, \Omega) = \frac{N_o \epsilon'_0}{2} \int_{-1}^1 d\mu \ \delta^3_{\mathrm{D}} \ = \ \Gamma \mathcal{E}' \ . \qquad (2.39)$$

The Lorentz boost simply adds a factor Γ to the total energy content, which is obvious by noting the symmetry of the transformation equation $\epsilon = \Gamma\epsilon'(1 + \beta\mu')$ with respect to μ'. But this example illustrates how the total energy can be calculated when dealing with more complicated angular distribuions of particles and photons.

2.7 Relativistic Reaction Rate

The relativistic reaction (or scattering) rate $\dot{n}_{sc} = dN/dV \, dt$ is defined as the number of collisions per unit volume per unit time between particle species "1" and "2" with masses m_1 and m_2, respectively. Because the number of collisions dN and the product $dV \, dt$ are separately invariant quantities, the ratio $\dot{n}_{sc}$ is also an invariant quantity.

Let the densities of species 1 and 2 in system K be denoted by n_1 and n_2, respectively, as shown in Fig. 2.3. Due to length contraction, the densities in the proper system in which the particles are at rest are given by $n_i = \gamma_i n_i^0$,

$i = 1, 2$, where $\gamma_i = (1 - \beta_i^2)^{-1/2}$ are the Lorentz factors of particles in K and we consider for the moment particles of type i that all move with the same Lorentz factor. Note that the density of particles is least in the proper frame. Transforming to the rest system of particle species 2 implies that the reaction rate in that system is

$$\dot{n}_{sc} = c\beta_r \sigma n_2^0 n_1' , \qquad (2.40)$$

where $c\beta_r$ is the relative speed of particles of type 1 in the rest system of particles of type 2, and $\sigma = \sigma(\gamma_r)$ is the scattering cross section. The quantity $\sigma(\gamma_r)$ is a function of $\gamma_r = (1 - \beta_r^2)^{-1/2}$, which is simply the relative Lorentz factor of a particle of one type in the rest system of the other particle type, and as defined is obviously invariant. From our preceding considerations, $\gamma_r = p_1^\mu \cdot p_2^\mu / (m_1 m_2) = \gamma_1 \gamma_2 (1 - \vec{\beta}_1 \cdot \vec{\beta}_2)$, where p_i^μ is the four-momentum of particles of type i (eq. [2.10]). Thus

$$\beta_r = \left[\frac{(p^1 \cdot p^2)^2 - m_1^2 m_2^2}{(p^1 \cdot p^2)^2} \right]^{1/2} . \qquad (2.41)$$

Note that $\beta_r = 1$ if either (or both) species are photons.

Let n_1' be the density of species 1 as seen in the rest system of species 2. Therefore $n_1' = \gamma_r n_1^0 = \gamma_r n_1 / \gamma_1$ and $n_2^0 = n_2 / \gamma_2$, implying $\dot{n}_{sc} = c\beta_r \sigma(\gamma_r)(1 - \vec{\beta}_1 \cdot \vec{\beta}_2) n_1 n_2$. This expression applies to two mono-energetic particle species each traveling in specific directions. In the general case, particles will have a distribution of directions and energies, so that it is necessary to integrate over the various directions and energies to calculate the total reaction rate. If the particle distributions are self-interacting, then the reaction rate must be multiplied by a factor of $1/2$ to correct for double counting. Thus the reaction rate for two interacting distributions of particles is given by

$$\dot{n}_{sc} = \frac{c}{(1 + \delta_{12})} \int \ldots \int \beta_r \, (1 - \vec{\beta}_1 \cdot \vec{\beta}_2) \sigma(\gamma_r) dn_1 dn_2 \qquad (2.42)$$

[4, 5], where $\delta_{12} = 1$ for self-interacting particle distributions and $\delta_{12} = 0$ for interactions of different types of particles. Because of the invariance of $\dot{n}_{sc}$, eq. (2.42) equally gives the reaction rate in frame K, even though it was derived in the proper frame of particle species 2. Eq. (2.42) is also valid for photon-particle interactions with $\beta_r \to 1$ and $\gamma_r \to \gamma_1 \epsilon (1 - \beta_1 \cos \theta_{12})$, and for photon-photon interactions with $\gamma_r \to \epsilon_1 \epsilon_2 (1 - \cos \theta_{12})$.

The differential spectral density

$$n(\vec{p}) = \frac{dn}{d^3 \vec{p}} = \frac{dn}{p^2 dp d\Omega} , \qquad (2.43)$$

so that $dn = n(\vec{p}) p^2 dp d\Omega$ and $d\Omega = d\mu d\phi$. The momentum of a particle or photon of species i is denoted by $\vec{p}_i$. For particles, $p_i = \beta_i \gamma_i$, whereas $p_i = h\nu_i / m_e c^2 = \epsilon_i$ for photons. The general expression for the reaction rate is therefore given by

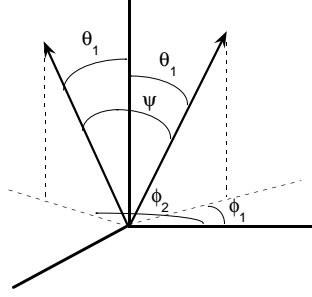$$\dot{n}_{sc} = \frac{c}{(1 + \delta_{12})} \oint d\Omega_1 \int_0^\infty dp_1 \, p_1^2 \, n_1(\vec{p}_1) \times$$

Figure 2.4: Angles in spherical geometry.

$$\oint d\Omega_2 \int_0^\infty dp_2 \; p_2^2 \; n_2(\vec{p}_2) \; \beta_r(1 - \beta_1\beta_2\cos\psi)\sigma(\gamma_r) \; . \tag{2.44}$$

The invariant energy defining the collision strength is the relative Lorentz factor $\gamma_r = \gamma_1\gamma_2(1 - \beta_1\beta_2\cos\psi)$, and ψ is the angle between the directions of the interacting particles or photons, given by

$$\cos\psi = \mu_1\mu_2 + (1 - \mu_1^2)^{1/2}(1 - \mu_2^2)^{1/2}\cos(\phi_1 - \phi_2) \tag{2.45}$$

(Fig. 2-4).

Now consider the scattering rate $\dot{N}_{sc}$ of a particle traversing a photon field. The distribution function for a single particle is $n_1(\vec{p}_1) = n_1\delta(p - p_1)\delta(\mu_1 - 1)\delta(\phi_1)/(4\pi p_1^2)$, and eq. (2.44) implies

$$\dot{N}_{sc}(p_1) = \frac{\dot{n}_{sc}}{n_1} = c \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \; (1 - \beta_1\mu) \int_0^\infty d\epsilon \; n_{ph}(\epsilon, \Omega) \; \sigma(\gamma_r) \; , \tag{2.46}$$

where $n_{ph}(\epsilon, \Omega) = dN/dV\,d\epsilon\,d\Omega$ is the photon distribution function and $\gamma_r = \gamma\epsilon(1 - \beta_1\mu)$ characterizes the invariant energy of the scattering event. For photon-photon ($\gamma\gamma$) interactions of a photon with energy ϵ_1 passing through a gas of photons with energy ϵ, $\beta_1 \to 1$ and $\gamma_r \to \epsilon_r = \epsilon\epsilon_1(1 - \mu)$. Thus the interaction rate of a photon with energy ϵ_1 passing through a background photon field $n(\epsilon, \Omega)$ is

$$\dot{N}_{\gamma\gamma}(\epsilon_1) = c \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \; (1 - \mu) \int_0^\infty d\epsilon \; n_{ph}(\epsilon, \Omega) \; \sigma(\epsilon_r) \tag{2.47}$$

2.8   Secondary Production Spectra

The calculation of secondary production spectra with momenta $\vec{p}_s = (p_s, \Omega_s)$ depends on knowledge of the differential cross section $d\sigma(p_1, \Omega_1, p_2, \Omega_2)/dp_s d\Omega_s$. In terms of differential densities, the results in the previous section imply the number emissivity

$$\dot{n}_s(p_s, \Omega_s) = \frac{dN_s}{dV\,dt\,dp_s\,d\Omega_s} = \frac{c}{1 + \delta_{12}} \oint d\Omega_1 \int_0^\infty dp_1 \; n_1(p_1, \Omega_1) \times$$

$$\times \oint d\Omega_2 \int_0^\infty dp_2 \; n_2(p_2, \Omega_2) \; \beta_r \cdot (1 - \beta_1\beta_2 \cos\psi) \; \frac{d\sigma(\gamma_r)}{dp_s d\Omega_s} \qquad (2.48)$$

for the seondary production spectra.

The most common type of problem encountered in black-hole studies involve scattering of photons by relativistic ($p \approx \gamma \gg 1$) particles, for example, in treatments of Compton scattering and photo-meson interactions. In this case, eq. (2.48) can be written as

$$\dot{n}_s(p_s, \Omega_s) = c \oint d\Omega \int_0^\infty d\epsilon \; n_{ph}(\epsilon, \Omega) \times$$

$$\times \oint d\Omega_{par} \int_1^\infty d\gamma \; (1 - \beta_{par}\cos\psi) \; n_{par}(\gamma, \Omega_{par}) \; \frac{d\sigma(\epsilon, \Omega, \gamma, \Omega_{par})}{dp_s d\Omega_s} \; . \qquad (2.49)$$

If the particle distribution function $n_{par}(\gamma, \Omega_{par})$ is assumed to be isotropic, then $n_{par}(\gamma, \Omega_{par}) = n_{par}(\gamma)/4\pi$.

The emissivity for electron-photon scattering is given by the expression

$$j(\epsilon_s, \Omega_s) = \frac{d\mathcal{E}}{dV\,dt d\epsilon_s d\Omega_s} = m_e c^2 \epsilon_s \dot{n}_s(\epsilon_s, \Omega_s) = c\epsilon_s \oint d\Omega \int_0^\infty d\epsilon \; \frac{u(\epsilon, \Omega)}{\epsilon} \times$$

$$\times \oint d\Omega_e \int_1^\infty d\gamma \; (1 - \beta_e \cos\psi) \; n_e(\gamma, \Omega_e) \; \frac{d\sigma(\epsilon, \Omega, \gamma, \Omega_e)}{d\epsilon_s d\Omega_s} \; , \qquad (2.50)$$

where the subscript "e" refers to the electron distribution. The specific spectral photon energy density $u(\epsilon, \Omega) = m_e c^2 \epsilon n_{ph}(\epsilon, \Omega)$ and scattering cosine angle

$$\cos\psi = \mu\mu_e + \sqrt{1-\mu^2}\sqrt{1-\mu_e^2}\cos(\phi - \phi_e) \; , \qquad (2.51)$$

from eq. (2.45).

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